Finite Duration Root Nyquist Pulses with Maximum In-Band Fractional Energy

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***Abstract*—We design root Nyquist pulses having maximum in-band fractional energy for a given finite bandwidth. The problem of maximizing the ratio of in-band energy to total energy has been dealt with earlier. But an exact solution could not be found since it involved optimization of a quadratic objective function with quadratic constraints. We have found an exact solution to this problem by taking a different approach. Numerical results are presented comparing the pulses so obtained with the raised cosine pulse and the better than raised-cosine (BTRC) pulse.**

***Index Terms*—Nyquist criteria, Nyquist pulses, root Nyquist pulses, optimization.**

I. Introduction

CONSIDER a pulse $g(t)$ satisfying the Nyquist criteria given by

$$g(t) = \begin{cases} 1 & t = 0; \\ 0 & t = kT, k = \pm 1, \pm 2, \ldots \end{cases} \tag{1}$$

where T is symbol duration. In a practical implementation, $g(t)$ is the convolution of the transmitter side pulse $h(t)$ and its matched filter $h^*(-t)$ at the receiver end i.e. $g(t) = h(t) \bigotimes h^*(-t)$ where $\bigotimes$ denotes convolution.

For several reasons, we require $h(t)$ to be of a small duration. The number of multiplication and addition operations that the receiver must perform per detected symbol are directly proportional to the pulse duration. In ISI channels, the number of interfering neighbors is small when the length of the pulse is small, which simplifies the task of equalization. The length cannot be made arbitrarily small however, since it would lead to poor confinement of the transmitted signal spectrum. This follows from the Uncertainty Principle [1], [2] which states that a signal cannot have arbitrary concentrations in the time and frequency domains simultaneously.

Therefore, in a practical implementation, the transmitter side band-limited pulse $h(t)$ is truncated in time domain. As a result of truncation, now its Fourier Transform (FT) has infinite support in frequency domain. We have limitations of channel bandwidth, so we need to concentrate most of its energy in a finite bandwidth to use the maximum spectral resources. For a given finite bandwidth, we wish to maximize the in-band fractional energy of the time-limited root Nyquist pulse $h(t)$ where in-band fractional energy can be defined as the ratio of the energy of the pulse in the given bandwidth to the total energy of the time-limited pulse. This problem has been previously dealt in [3] and [4].

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In [3], a cost function, based on Nyquist criteria, is set up which was quadratic as all of their calculations were based on root Nyquist pulse. An exact solution could not be found and the pulse obtained performed worse than BTRC in terms of in-band fractional energy. But, the paper introduced the idea of writing total energy and in-band energy in terms of eigenvalues of PSWFs. In [4], they tried to find a solution by defining a dual problem. All the calculations were again based on root Nyquist pulse which led to quadratic Nyquist constraints and hence, the final problem was not exactly solvable. Though it presented a detailed analysis of the optimization problem and proved the existence of a globally optimal solution, the approximate obtained solution of the problem is not pure Nyquist and has less in-band fractional energy than existing pulses RC and BTRC [5].

In Section II, the problem is formulated as expressing the overall Nyquist pulse in terms of PSWFs and an expression for the in-band fractional energy of the root Nyquist pulse is presented. The problem reduces to optimizing linear function with linear constraints and simplex method is used. In Section III, our approach is compared with the previous approaches. In Section IV, the in-band fractional energy of root Nyquist pulse is presented and compared with RC and BTRC . Finally, in Section V, conclusion and future work is discussed.

II. Problem Formulation

As discussed in Section I, we need to truncate the transmitter side pulse $h(t)$. There are two possible approaches for the truncation:

1) First truncate the Nyquist pulse $g(t)$ and then take the square root of the FT of the truncated $g(t)$ to find the FT of $h(t)$.
2) First take the square root of the FT of $g(t)$ and then truncate the resultant pulse in time domain to find $h(t)$.

In the second approach, there is no guarantee of getting the composite pulse as Nyquist pulse. Therefore we have taken the first approach in this paper and truncated the Nyquist pulse $g(t)$ outside a finite interval $[-T_p, T_p]$. So transmitter side pulse $h(t)$ is time-limited in $[-T_p/2, T_p/2]$ and we will refer to it as root Nyquist pulse. Note that the convolution of two time-unlimited pulses can result in a time-limited pulse. But since we are interested in only time-limited pulses, we confine ourselves to the set of time-limited root Nyquist pulses.

Let $g(t)$ denote overall Nyquist pulse time-limited to $[-T_p, T_p]$ and the considered band be $[-W, W]$ where $W = \frac{\pi(1+\alpha)}{T}$. For simplicity, we take $T = 1$. Now, we represent $g(t)$ as linear combination of Prolate Spheroidal Wave Functions.(see [1], [2] and [6]).

$$g(t) = \sum_{j=0}^{2N-2} a_j \psi_j(t) \tag{2}$$

where $j = 2m$, $m = 0, 1, 2, \ldots, N-1$, ψ_j is the PSWF of j^{th} order and N is the number of PSWFs used. If we take the combination of both even and odd PSWFs, we will have $2n+1$ constraints for Nyquist criteria as given by (1) for $2n$ symbol duration pulse. In case of taking only even PSWFs, if (1) is satisfied for positive values of k, then it gets automatically satisfied for negative values of k. Therefore, we will have only $n+1$ constraints and hence we could find an optimal solution using less number of PSWFs. Hence we have considered only even PSWFs.

Let $G(\omega)$ represents the FT of $g(t)$. Therefore, we have:

$$G(\omega) = \sum_j a_j (i)^j \sqrt{2\pi T_p \lambda_j / W} \psi_j(T_p \omega / W) \tag{3}$$

where $i = \sqrt{-1}$ and λ_j is the eigenvalue corresponding to ψ_j [7]. Let $H(\omega)$ represents the FT of $h(t)$. Therefore, we have:

$$G(\omega) = \mid H(\omega) \mid^2 \tag{4}$$

Then, the total energy of $h(t)$ is given by:

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mid H(\omega) \mid^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) d\omega \\ &= g(0) \end{aligned}$$

Since $g(t)$ is a Nyquist pulse, $g(0) = 1$. Thus the Nyquist criteria on $g(t)$ ensures unit energy constraint on the root Nyquist pulse. Hence to maximize in-band fractional energy, we need to just maximize the in-band energy. The in-band energy E_{ib} of a root Nyquist pulse, where the bandwidth of interest is $[-W, W]$, can be written as:

$$\begin{aligned} E_{ib} &= \frac{1}{2\pi} \int_{-W}^{W} G(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-W}^{W} \int_{-T_p}^{T_p} g(t) e^{-j\omega t} dt d\omega \\ &= \frac{1}{2\pi} \int_{-T_p}^{T_p} g(t) \int_{-W}^{W} e^{-j\omega t} d\omega dt \\ &= \int_{-T_p}^{T_p} g(t) \frac{sin(Wt)}{\pi t} dt \\ &= \sum_j a_j \int_{-T_p}^{T_p} \psi_j(t) \frac{sin(Wt)}{\pi t} dt \\ &= \sum_j a_j \psi_j(0) \lambda_j \end{aligned}$$

Hence, the problem reduces to:

$$\text{maximize} \quad \textstyle\sum_{j=0}^{2N-2} a_j \psi_j(0) \lambda_j \tag{5}$$

$$\text{subject to Nyquist criteria} \quad \textstyle\sum_{j=0}^{2N-2} a_j \psi_j(kT) = \delta_k \tag{6}$$

where δ_k is Kronecker delta and k is an integer such that $|kT| \leq T_p$. This ensures that the overall pulse is zero at the multiples of T excluding $k = 0$. This is an optimization problem involving a linear objective function and linear constraints, and hence can be solved using the simplex method.

III. Discussion

We have optimized the pulse in the sense of maximizing the in-band fractional energy of the root Nyquist pulse. Unlike [3] and [4], we have started our problem by expressing the Nyquist pulse as a linear combination of PSWFs. They had started by expressing a root Nyquist pulse as a linear combination of PSWFs as described below:

$$h(t) = \sum_j a_j \psi_j(t) \tag{7}$$

where $j = 2k$, $k = 0, 1, 2, \ldots$, $h(t)$ is the root Nyquist pulse. When Nyquist criteria is imposed on the above pulse, we have:

$$\int_{-T_p/2}^{T_p/2} h(t) h(t+kT) dt = \delta(k) \tag{8}$$

where k = 0,1,..., using (7) and (8), the Nyquist constraint reduces to:

$$\mathbf{a}^T B^{(k)} \mathbf{a} = \delta_k \tag{9}$$

where $\mathbf{a} = \{a_0, a_2, \ldots\}$ and $B_{ij}^{(k)} = \int_{-T_p/2}^{T_p/2} \psi_i(t) \psi_j(t + kT) dt$. Therefore, their approach leads to quadratic constraints. The expression for in-band fractional energy E_{ib} of root Nyquist pulse reduces to:

$$E_{ib} = \sum_j a_j^2 \lambda_j^2 \tag{10}$$

So the problem reduces to maximizing a quadratic objective function as given in (10) with quadratic constraints as given in (9). This problem is NP hard even for a pulse duration of one symbol. On the contrary, our approach leads to linear constraints as given in (6) and a linear objective function as given in (5), which is exactly solvable.

IV. Numerical Results

We have obtained the optimal pulse using (5) and (6). For a root Nyquist pulse with a duration of n symbols, we have $n+1$ constraints as given by (6). At least one more variable is needed to optimize for the in-band fractional energy. Therefore, we need at least $n+2$ unknowns to solve this problem and interestingly it achieves the optimal solution by using the minimum number of PSWF functions. Thus, for an optimal pulse for a duration of n symbols, an in-band fractional energy very close to unity is attained by using $N = n+2$ PSWF functions.

Let us consider a root Nyquist pulse of duration $n = 2$ symbols to be optimized for $\alpha = 0.25$. So we need $N = n+2$ i.e. 4 PSWFs to obtain the optimal root Nyquist pulse. The coefficients $a_j s$ obtained by solving (5) and (6) are $\{0.8589, -0.4142, 0.0355, 0.0482\}$.

Figure 1 shows the optimal root Nyquist pulse obtained and the root raised cosine (RRC) pulse for a pulse duration of 2 symbols and $\alpha = 0.25$. Both the pulses have unit energy in the time duration [-T,T]. As these are root Nyquist pulses, they need not be zero at the multiples of symbol duration. We found that the corresponding Nyquist pulses almost overlap.

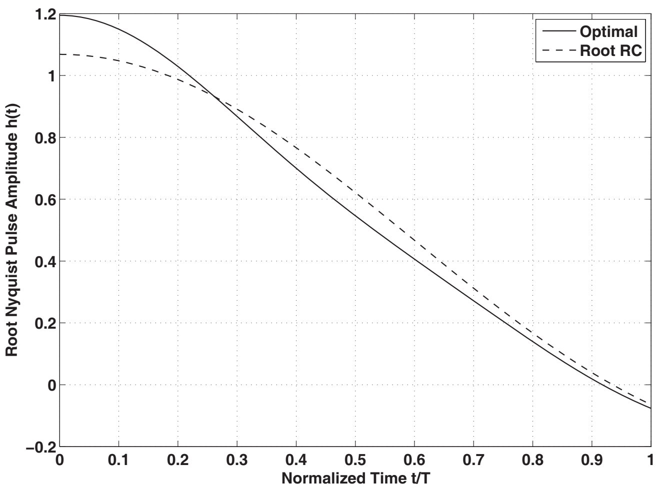

Fig. 1. 2 symbol duration optimal root Nyquist and root raise-cosine pulses with α = 0.25

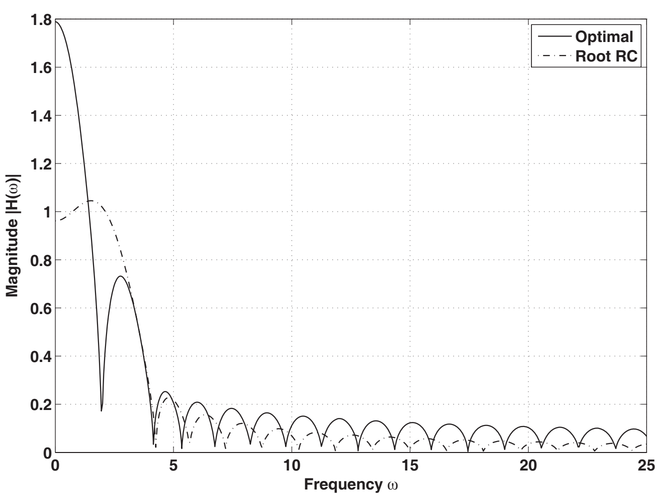

Fig. 2. FTs of 2 symbol duration optimal root Nyquist and root raise-cosine pulses with α = 0.25

Figure 2 has been obtained using (3) and (4). It shows the FTs of the optimal root Nyquist pulse and the RRC pulse for a pulse duration of 2 symbols and $\alpha = 0.25$. It shows that in the frequency domain the optimal pulse is more concentrated than RRC. The FT of root BTRC almost overlaps with that of RRC, so we have not shown it in the figure.

Figures 3 and 4 have been obtained by solving (5) and (6) and then computing the value of (5) for each α. We have varied the bandwidth $[-W, W]$ by varying α. These figures show the comparison of in-band fractional energy of optimal root Nyquist pulses with RRC and root BTRC pulses for pulse duration of 2 and 3 symbols respectively. In [3], pulses of 3 symbol duration have been given and we found that their in-band fractional energy lie between root RC and root BTRC pulses. These figures show that optimal root Nyquist pulses achieve an in-band fractional energy very close to 1. Though it is not exactly 1, but in the figure it is not distinguishable from 1. Also, they outperform all the existing pulses.

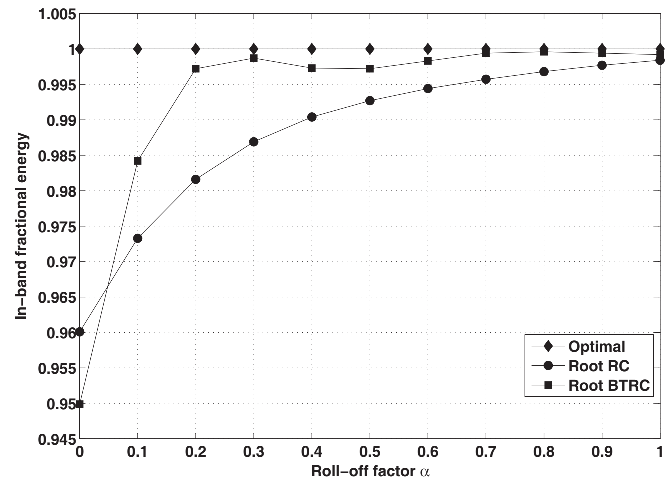

Fig. 3. Comparison of in-band fractional energy of optimal root Nyquist pulse with other root Nyquist pulses for 2 symbol duration for various α

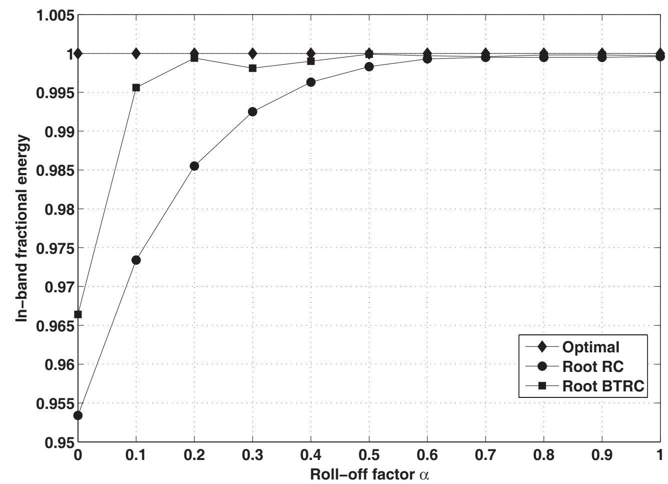

Fig. 4. Comparison of in-band fractional energy of optimal root Nyquist pulse with other root Nyquist pulses for 3 symbol duration for various α

V. Conclusion

A long standing problem of obtaining finite duration root Nyquist pulses which are optimal in the sense of achieving maximum in-band fractional energy at the transmitter end has been solved. Interestingly, in-band fractional energies very close to unity have been obtained.

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